Annals of "Dunarea de Jos" University of Galati
Fascicle I. Economics and Applied Informatics
Years XXXI – nº1/2025
ISSN-L 1584-0409 ISSN-Online 2344-441X
www.eia.feaa.ugal.ro

DOI https://doi.org/10.35219/eai15840409496

On the Tension Between Stability and Efficiency in Two-Way Flow Network with Small Decay

Banchongsan Charoensook[*]

ARTICLE INFO

Article history:
Accepted April 2025
Available online April 2025

JEL Classification
C72, D85

Keywords:
Network Formation, Nash Network, Two-way Flow Network, Agent Heterogeneity, Efficient Network

ABSTRACT

The tension between stability and efficiency in network formation models refers to a common finding in the literature of social networks: equilibrium networks and efficient networks often exhibit substantially different characteristics. In this note, I show that this tension can be partially reconciled in the two-way flow model of network formation with partner heterogeneity by introducing a small degree of information decay. This result extends a similar finding established for homogeneous agents in Charoensook (2025). Thus, this note contributes to the literature by demonstrating that the reconciliation of stability and efficiency through a small decay assumption is not confined to models with agent homogeneity but also holds in more general network settings.

1. Introduction

The study of network formation constitutes a rich, interdisciplinary field that leverages tools from game theory to understand how individuals or agents form networks by strategically creating connections. This approach is grounded in the premise that agents engage in a cost-benefit analysis when deciding whether to establish links with others. The motivation for this line of research stems from the observation that social networks play a critical role in shaping the diffusion of information across society, influencing important areas such as job search dynamics (Granovetter, 1974) and consumers' evaluations of products and innovations (Rogers & Kincaid, 1981).

Within this broad literature, the foundational contribution of Bala and Goyal (2000a) stands out. They introduced a formal model of network formation that is conceptually similar to a telephone call system: the initiator of a communication link incurs a cost to establish a channel through which both parties can exchange information, assumed to be nonrival. These links, once formed, allow bilateral information flows, effectively creating a two-way communication network.

Under the assumption of perfect information transmission—where information does not degrade as it travels through the network—Bala and Goyal (2000a) demonstrate that equilibrium networks exhibit remarkably simple structures. Only two types of networks remain stable in equilibrium: the empty network, in which no connections are formed, and the center-sponsored star, where a single central agent bears the cost of connecting to all others, thereby facilitating universal information access. However, such idealized conditions are rarely observed in actual social or economic networks.

One prominent deviation from perfect information flow is the assumption known as *small decay*. This refers to the realistic assumption that information slightly deteriorates as it travels through links in the network. The decay is sufficiently minimal that, if two agents are indirectly connected via intermediate nodes, the benefit of forming a direct link between them is outweighed by the cost. As a result, no agent has the incentive to pay for a shortcut connection. This concept is formally explored and given rigorous analytical treatment in the work of De Jaegher and Kamphorst (2015), and further extended in Charoensook (2020).

An important insight emerging from this line of inquiry is the strategic value of well-informed individuals within the network. Specifically, agents who receive a higher volume or quality of information from others—termed "best-informed" agents—become attractive targets for others to link with. These agents act as efficient hubs, providing superior information access compared to alternative potential connections. This understanding enables De Jaegher and Kamphorst (2015) to characterize the structural features of Nash networks, where each agent's link decisions form a pure-strategy Nash equilibrium.

Building upon these insights, Charoensook (2025) contributes a significant insight by demonstrating that best-informed agents not only serve as key nodes in equilibrium networks but are also the most effective

[*] Keimyung University, Daegu, South Korea. E-mail address: 11596@kmu.ac.kr

transmitters of information from an efficiency perspective. In other words, there is a structural overlap between Nash networks and efficient networks: the identity of the agent who receives links (i.e., the link receiver) is identical in both types of networks. Figure 1 and Proposition 1 in Charoensook (2025) offer a visual and formal articulation of this finding. This result presents a departure from the prevailing view in the literature, which generally finds that equilibrium networks (which are stable given agents' incentives) and efficient networks (which maximize overall welfare) diverge significantly in structure.

However, the result in Charoensook (2025) holds under a specific set of assumptions—namely, that all agents are homogeneous in terms of the cost they incur to form links and the value they place on the information they receive. This restricts the model's applicability and raises an important question: Can the equivalence between efficient and Nash link receivers persist in more general settings where agents are heterogeneous?

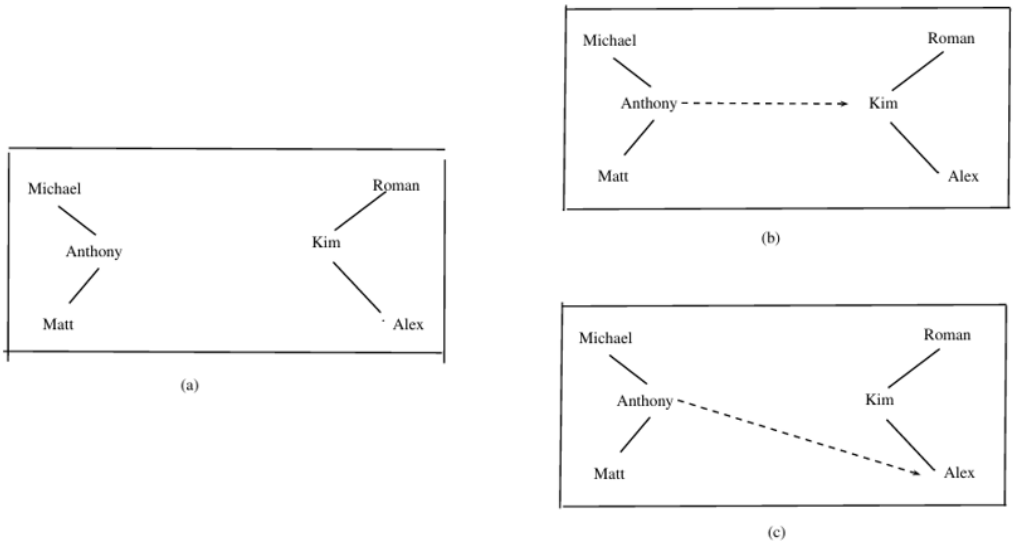

Figure 1.
On the left (Figure (a), two groups of agents remain disconnected. When the decay is small, within the group of Kim, Alex, and Roman, Kim, positioned at the center, holds more information than Alex and Roman (Lemma 1 in De Jaegher and Kamphorst (2015)). As a result, if Anthony (Anthony) seeks to acquire information from this group, he will strategically choose Kim as his connection, as illustrated in the upper right (Figure (b)), rather than Alex or Roman, as shown in the lower right (Figure (c)). Furthermore, Proposition 1 in Charoensook (2025) establishes that Kim (Kim) is also the most efficient agent for transmitting information within the network. This note extends that result to case of partner heterogeneity, where Anthony (Anthony) 's link formation cost and the value of information he receives depend on which of the three potential connections—Kim, Alex, or Roman —he chooses. This illustration is adapted from Figure 1 in Charoensook (2025)

In this note, I answer this question by extending the result of Charoensook (2025) to the case of *partner heterogeneity* in link formation cost and value, which is a form of agent heterogeneity that is well studied in the literature (see Charoensook (2020), Charoensook (2022), Billand et al. (2012), Olaizola and Valenciano (2021)). Partner heterogeneity refers to the assumption that the cost and/or benefit that an agent i incurs/receives from establishing a link with an agent j depends solely on the identity of agent j. As noted by Billand et al. (2011), introducing partner heterogeneity is a natural extension of Bala and Goyal (2000a)'s two-way flow network formation model, which originally assumes homogeneous players. Billand et al. (2011) also provide an illustration that is quoted as follows: "Consider Bala and Goyal (2000a)'s example of telephone communication: when making a call, the caller incurs a cost that varies depending on the recipient's identity. For instance, if the recipient is a particularly busy individual, reaching them may require greater effort or time, increasing the cost for the caller. Similarly, the value that a player derives from a connection is influenced by the information possessed by the recipient, which, in turn, is shaped by both their individual attributes and their position within the broader social network. Thus, network formation in heterogeneous settings naturally accounts for variations in costs and benefits based on the identities and characteristics of individuals involved". Proposition 1 in this note then shows that, given the small decay assumption and partner heterogeneity in link formation cost and value, every link receiver in a Nash network is an efficient link receiver. Hence, Proposition 1 in this note extends Proposition 1 in Charoensook (2025) to the case of partner heterogeneity. This extension is a major contribution of this work to the literature.

In relation to the broader literature, this finding carries significant implications. A core theme in the study of network formation is the inherent tension between equilibrium (or stability) and efficiency. Many prior works document that Nash networks, driven by individual incentives, often yield structures that are markedly different from those that would maximize overall social welfare (see Charoensook (2025) for a comprehensive review). The main contribution of Charoensook (2025) was to demonstrate that this tradeoff can be resolved in models that assume agent homogeneity and small decay, such as that of De Jaegher and

Kamphorst (2015). The present note deepens this insight by showing that the reconciliation between stability and efficiency can hold even when agents differ in the costs and values they associate with specific partners, which is a more realistic setting.

The remainder of this note is organized as follows. Section 2 introduces the formal model with partner heterogeneity. Section 3 presents the main theoretical results. Finally, Section 4 concludes by offering reflections on possible directions for future research.

2. The Model

The set of all agents in the network is N. We start by describing how agents form connections with one another. This process resembles making a phone call: an agent who initiates the connection incurs a cost to reach out to another agent. Once the connection is made, both agents share nonrival information. A link from agent i to agent j is denoted ij, where i is the initiator (or sender) and j is the recipient. The set of all potential links that agent i can initiate is represented by $L_i = \{ij \mid j \in N \backslash \{i\}\}$ and the set of all possible links in the network is $L \equiv \bigcup_{i \in N} L_i$.

A set of links formed by agent i, $g_i \subset L_i$, defines their individual *strategy*, and the combination of all agents' strategies,$g = \bigcup_{i \in N} g_i$, forms a *strategy profile*. The entire space of possible strategies is denoted by $G \equiv 2^L$.

If a link ij belongs to the strategy profile g, we say that agent i *accesses* agent j. The structure of g can be visualized as a *network*, where a directed arrow from i to j indicates that $ij \in g_i$. See Figure 2 in the next section for a graphical example.

Once a link between two agents is established, it enables the exchange of information, which is assumed to be nonrival and costless to transmit once the link exists. The framework follows the bidirectional (or "two-way") communication structure introduced in Bala and Goyal (2000a), whereby the existence of a link in either direction is sufficient for mutual information flow. Formally, we say that a link exists between agents i and j if $\overline{ij} \in g$, where $\overline{ij} \in g$ denotes that either $ij \in g$ or $ji \in g$. That is, the presence of a unidirectional link in either direction enables symmetric information access between the two agents.

Importantly, agents need not be directly linked to share information. If there exists a finite sequence of pairwise-connected agents that begins at i and ends at j, information can traverse this path and reach from i to j via intermediate nodes. Such a sequence is referred to as a *chain* in the network. Formally, a chain from i to j in a network g, denoted $P_{ij}(g)$, is defined as a sequence of agents $\{\overline{i_0 i_1}, \overline{i_1 i_2}, \dots, \overline{i_{k-1} i_k}\}$, with $i_0 = i$ and $i_k = j$, such that each consecutive pair $\overline{i_\ell i_{\ell+1}} \in g$ for $\ell = 0, \dots, k-1$. If such a path exists, agents i and j are said to be *connected* in the network.

Among all possible chains linking two agents, the one containing the smallest number of links is referred to as the *shortest chain*. The length of this chain provides a natural notion of separation between agents. Specifically, the *distance* from agent i to agent j, denoted $d_{ij}(g)$, is defined as the number of links in the shortest chain $P_{ij}(g)$. Following standard convention in the literature, the distance from any agent to itself is zero, i.e., $d_{ii}(g) = 0$ for all $i \in N$. Conversely, if there exists no such chain connecting i and j, we define the distance between them to be infinite, that is, $d_{ij}(g) = \infty$.

While agents are able to exchange information through both direct and indirect connections within a network, the quality or effectiveness of this communication is subject to attenuation. In particular, information transmission is not perfectly preserved as it traverses the network; rather, it degrades with each intermediate step due to frictional losses in the communication process. This attenuation is modeled using a constant geometric decay applied over the length of the communication path.

Formally, let $\sigma \in [0,1]$ denote the *decay factor*, which governs the rate at which information devalues as it passes through successive links. A decay factor of $\sigma = 1$ corresponds to perfect communication, where information is transmitted without loss regardless of the number of intermediaries. Conversely, values of $\sigma < 1$ imply that each link weakens the signal, capturing the diminishing reliability or relevance of information transmitted over greater distances.

Consider an agent j who possesses information of normalized value 1. If another agent i is connected to j via a path in the network g, and the length of the shortest such path—i.e., the minimal number of links—is given by $d_{ij}(g) = k$, then the effective value of the information received by i from j is given by σ^k. This formulation captures the idea that the further apart two agents are in the network, the less valuable the information one receives from the other becomes.

In the case where $i = j$, we define $d_{ii}(g) = 0$, and hence $\sigma^{d_{ii}(g)} = 1$, indicating that an agent fully retains their own information. If no chain exists between agents i and j, meaning $d_{ij}(g) = \infty$, we define $\sigma^\infty = 0$, implying that no information is transmitted between disconnected agents.

We now formalize the notion of the *small decay* assumption, a key assumption that simplifies the structure of equilibrium networks and aligns individual incentives with efficient network formation. This assumption plays a central role in determining whether agents find it beneficial to establish direct links as a means of improving access to information.

Consider a scenario in which agent j's information reaches agent i via an indirect path—i.e., a chain of multiple links involving intermediary agents. In this case, agent i may consider forming a direct link to agent j in order to shorten the communication path, thereby reducing the information decay. The potential benefit from this action is quantified by the improvement in information quality due to the reduction in path length, which translates to an increase in the decay-adjusted value of the information received. However, forming such a direct link incurs a non-negligible cost, associated with establishing and maintaining the connection.

The incentive for agent i to create this direct link depends on the relative magnitudes of the gain in information precision and the link formation cost. When the decay factor $\sigma \in [0,1]$ is sufficiently close to 1, the marginal improvement in information flow resulting from shortening the path becomes small. That is, if information hardly decays as it passes through the network (i.e., decay is "small"), then shortening a chain by one or more links yields only a negligible benefit. As a result, the payoff improvement does not justify the link formation cost, eliminating the incentive for unilateral link additions that aim to reduce path length.

From a system-wide or efficiency perspective, the same logic applies: when decay is sufficiently small, the aggregate informational gain to the network from adding redundant links becomes minor relative to the social cost of maintaining additional connections. Therefore, efficient networks under small decay tend to minimize redundancy, resulting in a minimal network where at most one path connects any pair of agents.

This analytical simplification is encapsulated in what is referred to as the *small decay assumption*, a condition studied in detail by De Jaegher and Kamphorst (2015). Throughout this note, we maintain this assumption as a standing condition. For formal implications of this assumption on network structure and equilibrium outcomes, the reader is referred to Lemma 1, which precedes Proposition 1 in the following section.

We now introduce a set of formal notations and definitions pertaining to network structure and information flow, which will be used throughout the remainder of the analysis.

Let $g \subseteq L$ be a network composed of links formed among agents in the finite set N. A *subnetwork* of g, denoted g', is any subset of links such that $g' \subseteq g$. A network g is said to be *connected* if, for every pair of distinct agents $i, j \in N$, there exists a path (or *chain*) of links in g through which information can flow from i to j. In other words, the network forms a single connected component.

A *component* of a network g is defined as a subnetwork $g' \subseteq g$ that is connected and is *maximal* with respect to set inclusion: no additional link from $g \backslash g'$ can be added to g' without violating its status as a distinct connected subset. Thus, components partition the network into disjoint, internally connected subnetworks.

A network is considered *empty* if it contains no links, i.e., $g = \emptyset$, and hence no communication or connection exists among any pair of agents. In contrast, a network or component is termed *minimal* if it remains connected, yet contains no redundant links—formally, if there exists at most one chain connecting any pair of agents. That is, the removal of any link from a minimal network would result in disconnection.

We also define agent-specific roles within a network. An agent $i \in N$ is said to be a *link sender* if there exists a link $xy \in g$ with $x = i$, and a *link receiver* if $y = i$ for some $xy \in g$. These designations reflect the directionality of link formation, particularly when costs are borne unilaterally by senders.

We now turn to notations relevant to the flow of information in minimal networks. Suppose g is a minimally connected network, so each pair of agents is connected via a unique chain. If a link $\bar{\imath}\bar{\jmath} \in g$ is removed, this action separates the network into two disconnected subnetworks. Let us denote by $D^i_{\bar{\imath}\bar{\jmath}}(g)$ the resulting subnetwork that contains agent i, and by $D^j_{\bar{\imath}\bar{\jmath}}(g)$ the one containing agent j. These are the two disjoint components created by the deletion of $\bar{\imath}\bar{\jmath}$, each of which is a connected subnetwork of $g \backslash \{\bar{\imath}\bar{\jmath}\}$.

Furthermore, let $N^i_{\bar{\imath}\bar{\jmath}}(g)$ and $N^j_{\bar{\imath}\bar{\jmath}}(g)$ denote the respective sets of agents belonging to the subnetworks $D^i_{\bar{\imath}\bar{\jmath}}(g)$ and $D^j_{\bar{\imath}\bar{\jmath}}(g)$. These sets will be instrumental in defining the notion of an *efficient link receiver* in later sections. For a graphical representation of these concepts, refer to Figure 2.

To formally describe how networks can be altered through local modifications, we introduce a notation for link addition and deletion. Let $g \subseteq L$ be a network consisting of directed links among agents in the finite set N. If $ij \in g$, then we denote the removal of this link by $g - ij$, which is shorthand for the set-theoretic difference $g \backslash \{ij\}$. This operation yields a new network identical to g except that the link from i to j is no longer present. Conversely, if $ij \notin g$, the operation $g + ij$ adds the directed link ij to the existing network, resulting in $g \cup \{ij\}$.

More generally, we allow for simultaneous modifications to the network. If one wishes to remove a link ij and introduce a new link kl, this is expressed as $g - ij + kl$, which is defined as the union $(g \backslash \{ij\}) \cup \{kl\}$. This operation enables comparisons of network structures, particularly when analyzing marginal changes in payoffs or welfare following a single link replacement.

Beyond individual link operations, we also formalize the notion of combining distinct subnetworks. Suppose g' and g'' are two disjoint subnetworks (i.e., they do not share any links or overlapping nodes) such that agent i belongs to the node set of g' and agent j belongs to the node set of g''. Then the operation $g' \oplus_{ij} g''$ denotes the new network formed by taking the union of g', g'', and the new bridging link ij, that is, $g' \oplus_{ij} g'' = g' \cup g'' \cup \{ij\}$. This notation captures the act of merging disconnected components through a single link, which is analytically useful in the study of minimal networks and efficient networks.

A special case of this operation arises in the context of minimally connected networks. Consider a network g and a link $ij \in g$ such that its removal partitions g into two disconnected subnetworks, denoted $D^i_{ij}(g)$ and $D^j_{ij}(g)$, containing agents i and j, respectively. Then the original network g can be reconstructed from these components by reintroducing the severed link, i.e., $D^i_{ij}(g) \oplus_{ij} D^j_{ij}(g) = g$. This identity will play an instrumental role in the proof of Lemma 2, which analyzes the welfare consequences of optimal reconnections following local deletions.

To capture individual-level heterogeneity in incentives and constraints across agents in a networked environment, we begin by formalizing two fundamental structures: the value that each agent derives from accessing information held by others, and the cost of initiating connections to obtain such information. Let V_{ij} denote the value agent i assigns to the information possessed by agent j, assuming the information is received without any loss or distortion. Collectively, these valuations define the information value structure $\mathcal{V} = \{V_{ij}\}_{i,j \in N}$, which may reflect complex asymmetries in the relative informativeness of agents from the perspective of others. In a particularly tractable setting, this structure satisfies partner heterogeneity: that is, the value derived from connecting to agent j is identical for all receivers, so $V_{ij} = V_j$ for every $i \in N$. This formulation treats the informativeness of each agent as an intrinsic characteristic, invariant across recipients.

In parallel, we define the cost structure $\mathcal{C} = \{c_{ij}\}_{i,j \in N, i \neq j}$, where c_{ij} represents the cost incurred by agent i to initiate a directed link to agent j. When $c_{ij} = c_j$ for all i, the structure also satisfies partner heterogeneity in costs: linking to a given agent imposes a cost determined solely by that agent's identity, regardless of the sender. The special case in which $V_{ij} = V > 0$ and $c_{ij} = c > 0$ for all distinct $i, j \in N$ defines a setting of agent homogeneity, where all agents are symmetric in both the informativeness they offer and the costs they impose on others.

Given these structures, the effective quantity of information transmitted in any realized network depends not only on the underlying valuations but also on the network structure and the attenuation of information across paths. Recall that $d_{ij}(g)$ is the length of the shortest path between agents j and i in g, and $\sigma \in [0,1]$ is the decay factor governing information degradation over distance, then the quantity of information received by agent i from agent j is given by

$$I_{ij}(g) = \sigma^{d_{ij}(g)} V_j.$$

When $i = j$, we assume $d_{ii}(g) = 0$, so $I_{ii}(g) = V_i$, reflecting the fact that agents retain their own information without decay. The total information accessible to agent i across the network is thus

$$I_i(g) = \sum_{j \in N} I_{ij}(g).$$

However, acquiring information requires incurring costs associated with forming direct connections. Let $N^S_i(g) = \{j \in N \mid ij \in g\}$ denote the set of agents to whom agent i sends links in the network g. We assume that link formation is costly and that these costs accumulate additively and are transformed by a strictly increasing cost function $C: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, capturing the disutility associated with establishing and maintaining multiple links. The utility or payoff of agent i in network g is then given by the difference between the total discounted information acquired and the total cost incurred:

$$U_i(g) = I_i(g) - C\left(\sum_{j \in N^S_i(g)} c_j\right). \quad [\text{Eq. 1}]$$

Let g^* represent a network, where agent i's strategy is given by $g^*_i \subset g^*$. Define $g^*_{-i} = g^* \backslash g^*_i$, so that the network can be written as $g^* = g^*_i \cup g^*_{-i}$.

Agent i's strategy g^*_i is a *best response* if it maximizes their utility, i.e.,

$$U_i(g^*) \geq U_i(g_i \cup g^*_{-i})$$

for all possible strategies g_i that agent i could choose.

A network g^* is called a *Nash network* if every agent in the network is playing a best response.

From a welfare perspective, it is of central interest to evaluate the aggregate payoff generated by a network. The total social welfare in network g is given by the sum of individual utilities:

$$W(g) = \sum_{i \in N} U_i(g).$$

Network g is said to dominate another network g' if $W(g) \geq W(g')$, and it is said to be efficient if it dominates all other possible networks. Substituting from the individual payoff expressions, we decompose welfare as

$$W(g) = \sum_{i \in N} I_i(g) - \sum_{i \in N} C\left(\sum_{j \in N_i^S(g)} c_j\right) = \bar{I}(g) - \bar{C}(g),$$

where

$$\bar{I}(g) = \sum_{i \in N} I_i(g), \quad \bar{C}(g) = \sum_{i \in N} C\left(\sum_{j \in N_i^S(g)} c_j\right).$$

These expressions also enable the characterization of which agents are most central to sustaining network efficiency. In a minimally connected network g, consider a link $\bar{xy} \in g$. Removing this link separates the network into two disjoint subnetworks. Let $N_{\bar{xy}}^y(g)$ denote the set of agents in the component containing y post-deletion. Suppose two agents $j', j'' \in N_{\bar{xy}}^y(g)$ are each considered as candidates for reconnecting to x to restore connectivity. Agent j' is said to be superior to j'' as a transmitter relative to link $\bar{xy}$ if:

$$W(g - \bar{xy} + \bar{xj'}) \geq W(g - \bar{xy} + \bar{xj''}).$$

If j' is superior to all other agents in $N_{\bar{xy}}^y(g)$, then j' is the efficient transmitter for $\bar{xy}$. An agent is an efficient link receiver (resp., sender) if they are the efficient transmitter for every link they receive (resp., send). Minimal efficient networks must exhibit this property across all agents.

The distribution of information in a network also determines which agents are best-informed. For any minimally connected subnetwork $M \subseteq N$, agent $i \in M$ is said to be *better-informed* than $j \in M$ if

$$\sum_{k \in M} I_{ik}(g) \geq \sum_{k \in M} I_{jk}(g),$$

and is the *best-informed* agent in M if this inequality holds for all $j \in M$. If $M = N_{\bar{xy}}^x(g)$ for some link $\bar{xy} \in g$, then i is the best-informed agent with respect to $\bar{xy}$.

It is essential to distinguish between the notions of best-informed agents and efficient link receiver. The former are identified based on the amount of information they personally receive, whereas the latter are characterized by the aggregate benefit they generate for the network when chosen as link recipients. Nonetheless, in the special case of agent homogeneity, these roles coincide: the best-informed agent is also the efficient transmitter. This equivalence is established by Charoensook (2025) and is extended to the case of partner heterogeneity in Proposition 1 of the following section.

3. Main Results

We begin by establishing two foundational lemmata that will form the basis for our subsequent characterizations of both equilibrium and efficient network structures. These results help clarify how the decay factor and heterogeneity in partner characteristics influence minimality and the identity of optimal link recipients. Our first lemma introduces the concept of *small decay*, which plays a critical role in ensuring that both Nash and efficient networks are minimal. The result identifies a threshold for the decay factor above which redundant links cease to be optimal, either from an individual or a social perspective.

Lemma 1. *[Adapted from Lemma 1 in Charoensook (2025)]*

Consider the payoff function defined in Equation (1), and suppose the value structure $\mathcal{V}$ and cost structure $\mathcal{C}$ satisfy partner heterogeneity. Then, for any link cost $c > 0$ and population size $n \geq 4$, there exists a decay threshold $\sigma_K < 1$ such that for all decay levels $\sigma > \sigma_K$, every nonempty Nash network and every nonempty efficient network is minimal.

The reasoning behind this result is intuitive. When information does not decay with distance (i.e., when $\sigma = 1$), shortening the path between two agents provides no marginal benefit since information is already perfectly transmitted. In such cases, forming additional links imposes a cost without yielding any informational gain. This logic extends to the setting where σ is close to one: when decay is sufficiently small, the marginal improvement in information access from shortening a path is minimal. Since the cost of link formation is positive and the improvement in information is negligible, agents have no incentive to introduce redundant connections. Continuity of the payoff function ensures that this conclusion holds for all σ sufficiently close to one. The minimality of Nash and efficient networks in this setting also follows from earlier results in Unlu (2018) and Billand et al. (2011).

We now state a second lemma, which formalizes how the total informational content of a network changes when two disconnected components are merged via a bridging link. This lemma will serve as a key tool in evaluating which agents should be chosen as link recipients to maximize overall information flow.

Lemma 2. *[Taken from Lemma 2 in Charoensook (2025)]*

Let g' and g'' be disjoint, minimally connected networks with agent sets N' and N'', respectively, and let $x \in N'$, $y \in N''$. Define $g = g' \oplus_{\bar{xy}} g''$, i.e., the network obtained by merging g' and g'' via the undirected link $\bar{xy}$. Then, the total information in g is expressed as: $\bar{I}(g) = \bar{I}(g') + \bar{I}(g'') + 2\sigma I_x(g')I_y(g'')$.

This identity reveals that the gain from connecting two components depends on both the decay factor and the informational centrality of the agents at the bridge endpoints. In particular, if one replaces a link $xy \in g$ with a link xz, and the receiver z is better-informed than y in the corresponding subnetwork. Formally, if $I_z\left(D^y_{xy}(g)\right) > I_y\left(D^y_{xy}(g)\right)$ then total information in the network increases so that $\bar{I}(g - xy + xz) > \bar{I}(g)$. This implies that, all else equal, agents should prefer to link with better-informed counterparts. Fixing the identity of the sender, the optimal recipient is the best-informed agent in the disconnected component.

This observation leads to the following proposition, which characterizes the structural properties of nonempty minimal networks that emerge under both Nash equilibrium and efficiency. The result provides a common condition on that every Nash network and every efficient network possesses concerning a link recipient.

Proposition 1.

Let the payoff function be as defined in Equation (1), and assume $n \geq 4$ and $\sigma \in (\sigma_K, 1)$. In any nonempty minimal Nash or efficient network g, the following two conditions hold for any link $xy \in g$:

1. *$c_y \leq c_{y'}$ for all $y' \in N^y_{xy}(g)$*
2. *For all $y'' \in N^y_{xy}(g)$ such that $c_y = c_{y''}$, we have $I_y\left(D^y_{xy}(g)\right) \geq I_{y''}\left(D^y_{xy}(g)\right)$*

   That is, every link receiver must be an efficient one.

Proof of the Proposition 1.

From Lemma 1, all Nash and efficient networks are minimal when σ is sufficiently close to 1. In the no decay case, a sender always links to the agent with the lowest cost. By continuity, this preference extends to the case of small decay. For the second property, Lemma 2 guarantees that linking to the best-informed agent maximizes total information in the efficient case. For Nash networks, the result follows from De Jaegher and Kamphorst (2015), who show that equilibrium link receivers must be the best-informed agents in their components. 0

This proposition generalizes Proposition 1 in Charoensook (2025), which included only the second condition. Under agent homogeneity, the first condition is automatically satisfied. However, in the presence of partner heterogeneity, it must be imposed explicitly to characterize equilibrium and efficient link choices.

Example 1.

In a minimal Nash network g as in the Figure 2(a), let $n = 10$ and $\sigma = 0.99$. Let agents consist of 2 groups: $\{L_1, L_2, L_3\}$, $\{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$ and set $c_{L_1} = c_{L_2} = c_{L_3} = 1.2$ and $c_{H_1} = c_{H_2} = \ldots = c_{H_7} = 1.5$ and $V_{L_1} = V_{L_2} = V_{L_3} = 100$ and $V_{H_1} = V_{H_2} = \ldots = V_{H_7} = 101$ where (To verify that this network is Nash, we simply need to verify that each link receiver is a best-informed agent and that any link removal, which lowers the link formation cost and at the same time lowers the informational quantity, does not lead to a higher payoff. Such a verification is straightforward yet tedious and hence is left to the readers to verify)*. Let the payoff be $U_i(g) = \sum_{j \in N \cup \{i\}} \sigma^{d_{ij}(g)} V_j - \left(\sum_{j \in N_i^s(g)} c_j\right)^2$. Consider the dotted link H_1L_1. Our first goal is to show that H_1's best response is to access either L_1, L_2 or L_3. First, observe that if there is no decay then certainly H_1's best response is to access either one of these three agents. If the decay is sufficiently small then by continuity H_1's best response is still to access one of these three agents depending on which one is best informed, since accessing other agents will incur excessive link formation cost. Specifically for $\sigma = 0.99$, we have that $I_{L_1}\left(D^{L_1}_{H_1L_1}(g)\right) = 883.02 > I_{L_2}\left(D^{L_1}_{H_1L_1}(g)\right) = I_{L_3}\left(D^{L_1}_{H_1L_1}(g)\right) = 882.07$ (see network $D^{L_1}_{H_1L_1}(g)$ in Figure 2(b)). Thus, H_1's best response is to access L_1.*

Next, let us check that H_1's decision to access L_1 rather than other agents maximizes total payoff of all agents in the network. That is, $W(g) > W(g - H_1L_1 + H_1z)$ for any agent $z \in N^{L_1}_{H_1L_1}(g)$. Again, first observe that if there is no decay then H_1's decision to access L_1, L_2 or L_3 rather than $H_2, H_3, \ldots, H_7$ leads to a higher total payoff due to a lower link formation cost. Now if we assume small decay then by the continuity of payoff this property continues to hold. It remains to be confirmed, therefore, that accessing L_1 rather than L_2 or L_3 leads to higher total amount of information. That is, $\bar{I}(g) > \bar{I}(g - H_1L_1 + H_1z)$ for every $z \in N^{L_1}_{H_1L_1}(g)$. Making use of Lemma 2 we *simply need to identify the best-informed agent among agents L_1, L_2 or L_3. Since, as shown in the previous paragraph, our best-informed agent is L_1. Thus, we conclude that H_1's action to access L_1 rather than other agents maximizes total payoff of all agents in the network. Finally, as in the previous paragraph, L_1 is also most attractive as a link receiver from the point of view of H_1. Hence, this example supports Proposition 1, which states that the identity of link receiver in a Nash network and the identity of link receiver in a Nash network is identical.*

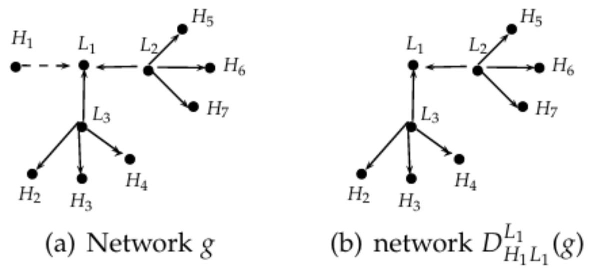

Figure 2. Example 1

4. Conclusions

In this note, I extend the main result of Charoensook (2025)—which states that every link receiver in a Nash network is an efficient link receiver—to the case where link formation costs and values exhibit partner heterogeneity. This extension contributes to the literature by demonstrating that the commonly observed tension between efficiency and stability in network formation models can be resolved not only in a specific case of agent heterogeneity but also in a broader setting of partner heterogeneity, which has been extensively studied in the literature.

As noted in Charoensook (2025), this result has the potential to address another challenge in the literature: the tendency for the set of efficient networks to be large and include networks with long diameters. In fact, under partner heterogeneity, Unlu (2018) shows that an efficient network can take the form of a line structure when there is no decay (see Section 4.1.3 in Unlu (2018)). An open question that remains is how to leverage this result to refine the set of efficient networks by incorporating the small decay assumption. I leave this question as a direction for future research.

References

1. *Bala, V., & Goyal, S. (2000a). A noncooperative model of network formation. Econometrica, 68(5), 1181–1229. https://doi.org/10.1111/1468-0262.00150*
2. *Bala, V., & Goyal, S. (2000b). A strategic analysis of network reliability. Review of Economic Design, 5(3), 205–228. https://doi.org/10.1007/s100580050018*
3. *Billand, P., Bravard, C., & Sarangi, S. (2011). Strict Nash networks and partner heterogeneity. International Journal of Game Theory, 40(3), 515–525. https://doi.org/10.1007/s00182-010-0253-4*
4. *Billand, P., Bravard, C., & Sarangi, S. (2012). On the interaction between heterogeneity and decay in two-way flow models. Theory and Decision, 73(4), 525–538. https://doi.org/10.1007/s11238-012-9292-7*
5. *Bloch, F., & Dutta, B. (2009). Communication networks with endogenous link strength. Games and Economic Behavior, 66(1), 39–56. https://doi.org/10.1016/j.geb.2008.03.010*
6. *Breitmoser, Y., & Vorjohann, P. (2013). Efficient structure of noisy communication networks. Mathematical Social Sciences, 66(3), 396–409. https://doi.org/10.1016/j.mathsocsci.2013.08.001*
7. *Charoensook, B. (2020). On the interaction between small decay, agent heterogeneity, and diameter of minimal strict Nash networks in two-way flow models. Annals of Economics and Finance, 21, 331–361.*
8. *Charoensook, B. (2022). A characterization of nonminimal Nash networks in two-way flow models. Annals of Economics and Finance, 23(2), 329–340.*
9. *Charoensook, B. (2025). On efficiency and stability in two-way flow networks with small decay: A note. ResearchGate. https://doi.org/10.13140/RG.2.2.17190.64323*
10. *De Jaegher, K., & Kamphorst, J. (2015). Minimal two-way flow networks with small decay. Journal of Economic Behavior & Organization, 109, 217–239. https://doi.org/10.1016/j.jebo.2014.11.013*
11. *Dutta, B., Ghosal, S., & Ray, D. (2005). Farsighted network formation. Journal of Economic Theory, 122(2), 143–164. https://doi.org/10.1016/j.jet.2004.03.007*
12. *Granovetter, M. (1974). Getting a job: A study of contacts and careers. University of Chicago Press.*
13. *Hojman, D. A., & Szeidl, A. (2008). Core and periphery in networks. Journal of Economic Theory, 139(1), 295–309. https://doi.org/10.1016/j.jet.2007.07.003*
14. *Jackson, M. O. (2007). Literature review: The study of social networks in economics. In J. E. Rauch (Ed.), The missing links: Formation and decay of economic networks. Russell Sage Foundation.*
15. *Jackson, M. O., & Wolinsky, A. (1996). A strategic model of social and economic networks. Journal of Economic Theory, 71(1), 44–74. https://doi.org/10.1006/jeth.1996.0108*
16. *Olaizola, N., & Valenciano, F. (2021). Efficiency and stability in the connections model with heterogeneous nodes. Journal of Economic Behavior & Organization, 189, 490–503. https://doi.org/10.1016/j.jebo.2021.07.016*
17. *Rogers, E. M., & Kincaid, D. L. (1981). Communication networks: Toward a new paradigm for research.*
18. *Unlu, E. (2018). Efficient networks in models with player and partner heterogeneity. The Manchester School, 86(1), 100–118. https://doi.org/10.1111/manc.12197*